

Springer Monographs in Mathematics

Ali Baklouti  
Hidenori Fujiwara  
Jean Ludwig

# Representation Theory of Solvable Lie Groups and Related Topics

 Springer

# Springer Monographs in Mathematics

## Editors-in-Chief

Minhyong Kim, School of Mathematics, Korea Institute for Advanced Study, Seoul, South Korea; Mathematical Institute, University of Warwick, Coventry, UK

Katrin Wendland, Research group for Mathematical Physics, Albert Ludwigs University of Freiburg, Freiburg, Germany

## Series Editors

Sheldon Axler, Department of Mathematics, San Francisco State University, San Francisco, CA, USA

Mark Braverman, Department of Mathematics, Princeton University, Princeton, NY, USA

Maria Chudnovsky, Department of Mathematics, Princeton University, Princeton, NY, USA

Tadahisa Funaki, Department of Mathematics, University of Tokyo, Tokyo, Japan

Isabelle Gallagher, Département de Mathématiques et Applications, Ecole Normale Supérieure, Paris, France

Sinan Güntürk, Courant Institute of Mathematical Sciences, New York University, New York, NY, USA

Claude Le Bris, CERMICS, Ecole des Ponts ParisTech Marne la Vallée, France

Pascal Massart, Département de Mathématiques, Université de Paris-Sud, Orsay, France

Alberto A. Pinto, Department of Mathematics, University of Porto, Porto, Portugal

Gabriella Pinzari, Department of Mathematics, University of Padova, Padova, Italy

Ken Ribet, Department of Mathematics, University of California, Berkeley, CA, USA

René Schilling, Institute for Mathematical Stochastics, Technical University Dresden, Dresden, Germany

Panagiotis Souganidis, Department of Mathematics, University of Chicago, Chicago, IL, USA

Endre Süli, Mathematical Institute, University of Oxford, Oxford, UK

Shmuel Weinberger, Department of Mathematics, University of Chicago, Chicago, IL, USA

Boris Zilber, Mathematical Institute, University of Oxford, Oxford, UK

This series publishes advanced monographs giving well-written presentations of the “state-of-the-art” in fields of mathematical research that have acquired the maturity needed for such a treatment. They are sufficiently self-contained to be accessible to more than just the intimate specialists of the subject, and sufficiently comprehensive to remain valuable references for many years. Besides the current state of knowledge in its field, an SMM volume should ideally describe its relevance to and interaction with neighbouring fields of mathematics, and give pointers to future directions of research.

More information about this series at <http://www.springer.com/series/3733>

Ali Baklouti • Hidenori Fujiwara • Jean Ludwig

# Representation Theory of Solvable Lie Groups and Related Topics

 Springer

Ali Baklouti  
Department of Mathematics  
University of Sfax  
Sfax, Tunisia

Hidenori Fujiwara  
Faculté de Science et Technologie pour  
l'Humanité  
Université de Kinki  
Iizuka, Japan

Jean Ludwig  
Institut Élie Cartan de Lorraine  
Université de Lorraine  
Metz, France

ISSN 1439-7382

ISSN 2196-9922 (electronic)

Springer Monographs in Mathematics

ISBN 978-3-030-82043-5

ISBN 978-3-030-82044-2 (eBook)

<https://doi.org/10.1007/978-3-030-82044-2>

© The Editor(s) (if applicable) and The Author(s), under exclusive license to Springer Nature Switzerland AG 2021

This work is subject to copyright. All rights are solely and exclusively licensed by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors, and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, expressed or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

This Springer imprint is published by the registered company Springer Nature Switzerland AG.  
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

# Preface

Representation theory of Lie groups and noncommutative harmonic analysis on Lie groups and homogeneous spaces have witnessed a significant growth starting from the 1940s when research work in these areas began. Taken jointly, they constitute a pivotal and compelling field within mathematics, due to the tight relationships with so many other areas—think of number theory, algebraic geometry, differential geometry, operator algebras, PDEs, to name but a few—and also with farther fields like physics.

Within the setting of exponential solvable Lie groups, the so-called orbit method is a fundamental tool to associate through an elegant way unitary duals to the space of coadjoint orbits. Still, the harmonic analysis on the corresponding homogeneous spaces remains a difficult subject.

The purpose of this book is to discuss the latest advances in this area, prove novel results on noncommutative harmonic analysis on solvable homogeneous spaces, and provide many applications. The text offers the most recent solutions to a number of open questions posed over the last decades, presents the newest research results on the matter, and provides an alluring platform for progressing in this research area.

Throughout the text, unless otherwise explicitly stated,  $G$  will always denote a connected and simply connected exponential solvable Lie group with Lie algebra  $\mathfrak{g}$ . This means the exponential mapping  $\exp : \mathfrak{g} \rightarrow G$  is a diffeomorphism. The first chapter aims to build the branching laws of *mixed representations* (or representations of mixed type), showing that they obey the orbital spectrum formula. Let  $A$  and  $H$  be two closed connected subgroups of  $G$ , and  $\pi$  and  $\sigma$  two unitary irreducible representations of  $G$  and  $H$ , respectively. Mixed representations are unitary representations of the form  $(\text{ind}_H^G \sigma)|_A$  (up-down representation) or  $\text{ind}_H^G(\pi|_H)$  (down-up representation). We shall provide explicit formulas for the multiplicity function occurring in the disintegration of such representations, and detailed information regarding its behavior. We shall study many situations where the multiplicity function is uniformly infinite or finite and bounded, and discuss along the way necessary and sufficient conditions for finiteness. One major motivation for this work goes back to the theory of ergodic Lie group actions. Another is

the need to build smooth disintegrations of the space  $L^2(G)$  on a Lie group  $G$  from a representation-theoretical point of view.

Now, one of the most important and difficult problems in the disintegration theory of group representations is to write down explicit unitary operators that intertwine unitary representations and their decomposition into irreducibles. This is, as a matter of fact, the basic aim of Chap. 3. The answer to this question can help to solve several important problems, for example regarding the study of the solvability of invariant differential operators on homogeneous spaces, or the Frobenius reciprocity problem related to the corresponding multiplicities, the disintegration of generalized invariant vectors, etc. We shall construct intertwining operators of induced representations (sometimes concerning discrete subgroups) and restricted representations of arbitrary nilpotent Lie groups. This will elicit the analysis of the tensor product of two unitary representations. Even in the case of exponential solvable Lie groups these problems are still not fully solved, and we shall only discuss the case of induced representations from normal subgroups.

With the above in mind, we establish the Bonnet-Plancherel formula associated with a monomial representation of a nilpotent Lie group and, more generally, of an exponential solvable Lie group when the inducing subgroup is normal. In Chap. 4, we will also discuss a variant of the Penney-Plancherel formula for induced representations and restrictions.

The orbit method of Chap. 2, which goes back to the early 1970s, asserts that two monomial representations  $\pi_i = \text{ind}_{H_i}^G \chi_f (H_i = \exp(\mathfrak{h}_i), 1 \leq i \leq 2)$  of  $G$  are irreducible and mutually equivalent. Here  $\mathfrak{h}_1$  and  $\mathfrak{h}_2$  are two polarizations of  $\mathfrak{g}$  at  $f \in \mathfrak{g}^*$  that satisfy the Pukanszky condition. We construct an explicit intertwining operator between these representations in the absence of a complete proof of the fact that the product  $H_1 H_2$  is a closed subset of  $G$ . A third Vergne polarization will come into play and give rise to a unitary intertwining operator. We shall prove a composition theorem involving the Maslov index of the three polarizations. The fundamental results of Auslander-Kostant [3] and Pukanszky [144] about holomorphically induced representations of solvable Lie groups, which hold without any knowledge of real polarizations for a given linear form in the dual Lie algebra, live outside the world of exponential Lie groups. There clearly remains a massive number of open questions to be solved. Chapter 6 will provide the occasion to study certain intertwining operators and real polarizations in the aforementioned rather general setting.

The aim of Chap. 5 is to present two open conjectures concerning certain algebras of invariant differential operators on the space of induced and restricted representations of nilpotent Lie groups. The claim is that whenever these representations have finite multiplicities, the corresponding algebras are isomorphic to the algebra of invariant polynomial functions on some related geometric object (manifold). We will furnish solutions in many special cases and lay out the latest advances in this direction.

Chapter 7 is a sort of generalization of Chaps. 2 and 4 to the setting of exponential groups  $G$ . We shall study in every detail the finite and infinite multiplicity of

discrete-type monomial representations. Our interest lies in the Penney-Plancherel formula for monomial representations of discrete type, and in a question of Michel Duflo (cf. [54]) regarding the commutativity of the algebra of  $G$ -invariant differential operators on line bundles associated with the representation in question. We shall emphasize the result whereby such algebra turns out to be commutative in the case of monomial representations of discrete type and hence gives a counterexample to Duflo's question. The orbit method (cf. [70]) will be a fundamental technique in the study.

The last chapter (Chap. 8) is an application of earlier results of the book. We will characterize bounded, topologically irreducible representations of an exponential solvable Lie group  $G$  on Banach spaces by using a triple involving topologically irreducible weighted representations. We shall determine the simple modules of the group algebra  $L^1(G)$ . These are essentially obtained in the same way as for nilpotent groups, except that one has to generalize the induced representations. This is no longer true for topologically irreducible representations, as shown in [117]. We will show that if the group  $G$  is not symmetric, a new type of bounded, irreducible representation of  $G$  on Banach spaces crops up. The corresponding representation of  $L^1(G)$  is fundamentally different from the induced representation. The construction of such representations is related to the invariant-subspace problem.

Sfax, Tunisia  
Iizuka, Japan  
Metz, France

Ali Baklouti  
Hidenori Fujiwara  
Jean Ludwig



# Nomenclature

$\rho(G, H, A, \sigma)$	up-down representation
$\mathcal{E}(G, K)$	the space of continuous functions $\xi$ on $G$ with compact support modulo $K$
$(\mathcal{H}_\rho^{-\infty})^{K,\lambda}$	space of $K$ -semi-invariant generalized vectors
$\beta_\phi$	Penney distribution
$\exp : \mathfrak{g} \rightarrow G$	The exponential mapping
$\lambda_G$	left regular representation
$\mathbb{C}(\Omega)^K$	field of invariant rational functions
$\mathbb{C}(\Gamma_\tau)^H$	field of invariant rational functions
$\mathbb{C}[\Omega(\pi)]^K$	algebra of invariant polynomial functions
$\mathbb{C}[\Gamma_\tau]^H$	algebra of invariant polynomial functions
$\mathfrak{b}$	real kernel part
$\mathfrak{d}$	real part
$\mathfrak{e}$	complex part
$\mathfrak{g}^*/G$	space of coadjoint orbits
$\mathcal{S}(N)$	the Schwartz-algebra of $N$
$\mathcal{C}$	composition sequence
$\mathcal{H}(f, \mathfrak{h}, N)$	space of holomorphically induced representation
$\mathcal{H}(f, \eta_f, \mathfrak{h}_i, G)$	space of holomorphically induced representation
$\mathcal{S}$	good sequence of subalgebras
$\mathcal{U}(\mathfrak{g}, \tau)$	infinitesimal invariant operator
$\omega_U$	fundamental weight
$\pi \times \pi'$	outer tensor product of two representations
$\pi _H$	a restriction of a representation to a subgroup
$\rho(f, \mathfrak{h}, N)$	holomorphically induced representation
$\rho(f, \eta_f, \mathfrak{h}, G)$	holomorphically induced representation
$\rho(G, H, \pi)$	down-up representation
$\text{ind}_H^G \nu$	induced representation
$\Delta_G$	modular function
$\Theta$	algebra homomorphism
$\Theta_G$	the Kirillov-Bernat mapping

$\widehat{A}^{\text{top}}$	equivalence classes of irreducible Banach space modules of $A$
$a(\mathfrak{h}_i, \mathfrak{h}_j, \mathfrak{h}_k)$	$= e^{\frac{i\pi}{4}\tau_{ijk}}$
$a_\ell$	pointwise Penney distribution
$a_\pi^k$	Penney's distribution
$c(i, j, k)$	Maslov index
$D_\pi(G)^K$	algebra of invariant differential operator
$D_\tau(G/H)$	algebra of invariant differential operators
$G \cdot l$	coadjoint orbit
$H \backslash G / B$	a double coset
$H(x, y)$	Hermitian form
$H_f(x, y)$	Hermitian form
$I_{\mathfrak{h}_2 \mathfrak{h}_1}$	formal intertwining integral
$J$	complex structure
$K(s, m)$	integral
$P^+ = P^+(f, \mathfrak{n})$	set of positive polarizations
$P_X$	image function
$R_{a, G/P, \ell}$	retract
$S(e)$	set of jump indices
$S(x, y)$	symmetric form
$\text{Simple}(A)$	equivalence classes of simple modules
$T(e)$	set of non-jump indices
$T(f, \eta_f, \mathfrak{h}, G)$	induced representation
$T_{\mathfrak{h}_2 \mathfrak{h}_1}$	intertwining operator
$U_e$	layer
$X^{\text{fin}}$	finite-rank subspace
$\delta_\ell(s) = \delta_{\ell, \mathcal{S}}(s)$	weight factor
$\mathfrak{a}_\tau$	kernel
$\nu_{\mathfrak{h}}(s)$	action of an automorphism
$\tau_{ijk}$	Maslov index
$\theta_{\mathfrak{h}}(s)$	density
$\Lambda_{\mathfrak{h}}$	intertwining isomorphism

# Contents

<b>1</b>	<b>Branching Laws and the Multiplicity Function of Unitary Representations of Exponential Solvable Lie Groups</b>	1
1.1	Introduction	1
1.2	Generalities and Notations	2
1.2.1	Coexponential Bases	2
1.2.2	Modular Functions and Quotient Measures	3
1.2.3	Induced Representations	5
1.2.4	Polarizations	5
1.2.5	Orbit Theory	6
1.2.6	Branching Laws: Induced Representations	6
1.2.7	Restrictions	7
1.3	Pseudo-Algebraic Geometry	7
1.3.1	Pseudo-Algebraic Sets	7
1.3.2	Semi-analytic Sets	10
1.3.3	Structure of Coadjoint Orbits	10
1.4	Up-Down Representations of Exponential Solvable Lie Groups	11
1.4.1	Disintegration of Up-Down Representations	12
1.4.2	The Multiplicity Function of Up-Down Representations	15
1.5	Down-Up Representations	24
1.5.1	The Down-Up Formula	25
1.5.2	The Down-Up Multiplicity Formula	28
1.5.3	Examples	30
1.5.4	The Case of Exponential Solvable Groups	33
1.6	The Multiplicity Function of Monomial Representations	37
<b>2</b>	<b>Intertwining Operators for Irreducible Representations of an Exponential Solvable Lie Group</b>	43
2.1	Introduction	43
2.2	A Trace Relation	43
2.3	Relations Between Two Polarizations	52

2.4	Vergne Polarizations .....	61
2.5	The General Case .....	64
2.6	A Local Result .....	67
2.7	The Case Where $\mathfrak{h}_1 + \mathfrak{h}_2$ Is a Subalgebra .....	93
2.8	The Key Point Is the Convergence .....	96
<b>3</b>	<b>Intertwining Operators of Induced Representations and Restrictions of Representations of Exponential Solvable Lie Groups</b> .....	<b>107</b>
3.1	Introduction .....	107
3.2	Intertwining Operators of Induced Representations of Nilpotent Lie Groups .....	107
3.2.1	Generalities .....	109
3.2.2	Disintegration of Monomial Representations .....	110
3.2.3	Construction of the Intertwining Operator .....	117
3.2.4	Examples .....	137
3.3	The Case of Exponential Solvable Groups .....	141
3.3.1	A Base Space of the Disintegration of Induced Representations .....	142
3.3.2	Construction of the Intertwining Operator .....	144
3.3.3	The Inverse Operator .....	151
3.3.4	A Rational Disintegration of $L^2(G)$ for an Exponential Solvable Lie Group $G$ .....	156
3.3.5	Examples .....	157
3.4	Intertwining of Representations Induced from Maximal Subgroups of Exponential Solvable Lie Groups .....	160
3.5	Intertwining Operators of the Restriction of Representations of Nilpotent Lie Groups .....	171
3.5.1	Double-Coset Space .....	171
3.5.2	A Measure on $H \backslash G / B$ .....	182
3.5.3	A Concrete Intertwining Operator .....	189
3.6	Disintegrating Tensor Products of Irreducible Representations of Nilpotent Lie Groups .....	191
3.6.1	A Concrete Intertwining Operator for Tensor Products of Unitary Representations .....	191
3.6.2	A Concrete Example .....	197
3.6.3	Criteria for Irreducibility of Tensor Products .....	199
3.7	Intertwining of Quasi-Regular Representations of Nilmanifolds .....	202
3.7.1	Rational Structures and Uniform Subgroups .....	203
3.7.2	Intertwining Operators .....	206
3.7.3	On the Multiplicity Formula .....	216
3.7.4	Primary Projections .....	219
3.7.5	Characterization of Two-Step Nilmanifolds with Equivalent Quasi-Regular Representations .....	221

3.7.6	Decomposition of the Quasi-Regular Representation $R_\Gamma$ .....	222
3.7.7	Intertwining Operators .....	225
<b>4</b>	<b>Variants of Plancherel Formulas for Monomial Representations of Exponential Solvable Lie Groups</b> .....	<b>235</b>
4.1	Layout of the Problems .....	235
4.2	The Penney-Plancherel Formula for Nilpotent Lie Groups .....	236
4.2.1	Tempered Distributions of Positive Type .....	237
4.2.2	Well-Adapted Bases .....	239
4.3	The Plancherel-Bonnet Formula for Normal Inducing Subgroups of Exponential Solvable Lie Groups .....	243
4.3.1	$G$ -equivariant Projections .....	244
4.3.2	Sobolev Spaces .....	246
4.3.3	Sobolev Spaces and Monomial Representations .....	248
4.3.4	Polarizations .....	250
4.3.5	Decomposition of Measures .....	251
4.3.6	The Bonnet-Plancherel Formula .....	254
4.3.7	A Variant of Penney's Plancherel Formula .....	261
4.4	The Penney-Plancherel Formula for Finite-Multiplicity Restrictions of Nilpotent Lie Groups .....	265
4.4.1	On Restrictions of Unitary Representations .....	266
4.4.2	The Plancherel and Penney-Plancherel Formulas .....	267
4.4.3	Examples .....	268
4.4.4	Proof of the Main Results .....	273
4.4.5	The Case of Normal Subgroups .....	277
4.4.6	An Intertwining Operator .....	278
<b>5</b>	<b>Polynomial Conjectures</b> .....	<b>281</b>
5.1	Introduction .....	281
5.2	The Case of Induced Representations .....	282
5.2.1	Towards the Conjecture .....	285
5.2.2	Special Cases .....	292
5.3	The Case of Restricted Representations .....	309
5.3.1	Frobenius Vectors .....	311
5.3.2	The Function $P_W$ on $\Omega(\pi)$ .....	314
5.3.3	Further Study of the Conjecture .....	320
5.3.4	Case 1. $\mathfrak{h} \subset \tilde{\mathfrak{n}}$ .....	327
5.3.5	Case 2. $\mathfrak{h} \not\subset \tilde{\mathfrak{n}}$ .....	327
5.3.6	Examples .....	341
<b>6</b>	<b>Holomorphically Induced Representations of Solvable Lie Groups</b> ...	<b>351</b>
6.1	Introduction .....	351
6.2	Intertwining Operators .....	351
6.2.1	Nilpotent Lie Groups and Maslov Index .....	352
6.2.2	Study of Connected Solvable Lie Groups .....	363

6.2.3	Explicit Expression of Intertwining Operators .....	370
6.2.4	Examples .....	377
6.3	Real Polarizations.....	378
6.3.1	Preliminaries .....	379
6.3.2	Irreducibility and Equivalence .....	386
<b>7</b>	<b>Monomial Representations of Discrete Type of Exponential Solvable Lie Groups .....</b>	<b>401</b>
7.1	Introduction .....	401
7.2	Preliminaries .....	401
7.3	Monomial Representations of Discrete Type.....	403
7.3.1	Generic and Strongly Generic Elements .....	405
7.3.2	A Basis for $\mathfrak{h}/(\mathfrak{h} \cap \mathfrak{b})$ .....	412
7.4	A Convergence Proof.....	423
7.5	The Concrete Plancherel Formula .....	430
7.6	Invariant Differential Operators .....	444
7.7	Polarizations .....	449
<b>8</b>	<b>Bounded Irreducible Representations .....</b>	<b>453</b>
8.1	Introduction .....	453
8.2	Simple Modules of Banach Algebras .....	455
8.2.1	Elementary Definitions .....	455
8.2.2	The Spectrum in Banach Algebras .....	462
8.2.3	Simple Modules and the Spectrum .....	465
8.2.4	Construction of Simple Modules .....	467
8.2.5	Simple $A$ -modules and Simple $pAp$ -modules .....	470
8.3	Irreducible Banach-Space Representations and Projections .....	487
8.3.1	Submodules of an Irreducible Module .....	487
8.3.2	Minimal Norm and Extension Norms.....	490
8.3.3	Topologically Simple Norms .....	492
8.4	Restricting and Extending Ideals.....	494
8.4.1	Definitions .....	494
8.4.2	Description of Extended and Restricted Ideals .....	496
8.5	Polynomial Growth and Functional Calculus .....	499
8.5.1	Definitions and Elementary Properties.....	499
8.5.2	Principles of Functional Calculus .....	504
8.5.3	Estimate for $\ u(nf)\ _{\omega}$ .....	505
8.5.4	Properties of Functional Calculus .....	505
8.5.5	Computation of the Bound Used in Functional Calculus .....	506
8.6	Simple Modules of $L^1(G)$ for Nilpotent Lie Groups .....	511
8.7	Fell's Topology on $Prim(G)$ and the Wiener Property.....	519
8.8	Variable Groups.....	525
8.8.1	Kirillov's Conjecture for Nilpotent Lie Groups .....	531
8.8.2	Coefficients of Monomial Representations .....	535

- 8.9  $\mathcal{D}$ -prime Ideals in the Schwartz Algebra of a Nilpotent Lie Group ..... 539
  - 8.9.1 Exponential Actions ..... 539
  - 8.9.2 Proof of Theorem 8.9.6 ..... 541
- 8.10 A Retract Theorem ..... 545
  - 8.10.1 Smooth Kernels ..... 549
  - 8.10.2 A Retract Theorem for Exponential Orbits in a Nilpotent Lie Group's Spectrum ..... 554
  - 8.10.3 An Application ..... 558
- 8.11 Bounded Irreducible Representations of  $G$  ..... 564
  - 8.11.1  $G$ -prime Ideals ..... 564
  - 8.11.2 The Representation  $\pi^\gamma = \text{ind}_H^G \gamma$  ..... 567
  - 8.11.3 An Example: Representations on Mixed  $L^p$ -spaces ..... 569
  - 8.11.4 The Spaces  $\mathcal{E}S$  ..... 573
- 8.12 Using Projections in  $L^1(G)/\ker_{L^1(G)}(\pi^\gamma)$  ..... 576
  - 8.12.1 The Weight  $\omega$  ..... 579
  - 8.12.2 The Algebras  $(p_\lambda * L^1(G) * p_\lambda)/\ker_{L^1(G)}(\pi^\lambda)$  and  $L^1(S, \omega)$  ..... 580
  - 8.12.3 Conclusion: Two Problems to Solve ..... 582
- 8.13 Irreducible Representations of  $L^1(S, \omega)$  ..... 583
  - 8.13.1 Characters and Other Examples of Irreducible Representations ..... 583
  - 8.13.2 Estimating the Weight  $\omega$  ..... 584
- 8.14 Classifications of Bounded Irreducible  $G$ -modules: The Main Theorem ..... 590
  - 8.14.1 Relationships Between Kernels ..... 591
- 8.15 Characterization of Simple Modules of  $L^1(G)$  ..... 592
  - 8.15.1 A Family of Simple Modules ..... 592
  - 8.15.2 A Character ..... 592
  - 8.15.3 Analysis of Simple  $L^1(G)$ -modules ..... 595
  - 8.15.4 Equivalence of Two Simple  $L^1(G)$ -modules ..... 597
  - 8.15.5 Symmetric Group Algebras ..... 600
  - 8.15.6 Final Remarks ..... 601
- Bibliography** ..... 603
- Index** ..... 609