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Representation Theory of Solvable Lie Groups and Related Topics

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Preface

Representation theory of Lie groups and noncommutative harmonic analysis on Lie groups and homogeneous spaces have witnessed a significant growth starting from the 1940s when research work in these areas began. Taken jointly, they constitute a pivotal and compelling field within mathematics, due to the tight relationships with so many other areas—think of number theory, algebraic geometry, differential geometry, operator algebras, PDEs, to name but a few—and also with farther fields like physics.

Within the setting of exponential solvable Lie groups, the so-called orbit method is a fundamental tool to associate through an elegant way unitary duals to the space of coadjoint orbits. Still, the harmonic analysis on the corresponding homogeneous spaces remains a difficult subject.

The purpose of this book is to discuss the latest advances in this area, prove novel results on noncommutative harmonic analysis on solvable homogeneous spaces, and provide many applications. The text offers the most recent solutions to a number of open questions posed over the last decades, presents the newest research results on the matter, and provides an alluring platform for progressing in this research area.

Throughout the text, unless otherwise explicitly stated, G will always denote a connected and simply connected exponential solvable Lie group with Lie algebra $\mathfrak{g}$. This means the exponential mapping $\exp : \mathfrak{g} \longrightarrow G$ is a diffeomorphism. The first chapter aims to build the branching laws of *mixed representations* (or representations of mixed type), showing that they obey the orbital spectrum formula. Let A and H be two closed connected subgroups of G, and $\pi and \sigma$ two unitary irreducible representations of G and H, respectively. Mixed representations are unitary representations of the form $(\mathrm{ind}_H^G \sigma)|_A$ (up-down representation) or $\mathrm{ind}_H^G(\pi_{|H})$ (down-up representation). We shall provide explicit formulas for the multiplicity function occurring in the disintegration of such representations, and detailed information regarding its behavior. We shall study many situations where the multiplicity function is uniformly infinite or finite and bounded, and discuss along the way necessary and sufficient conditions for finiteness. One major motivation for this work goes back to the theory of ergodic Lie group actions. Another is the need to build smooth disintegrations of the space $L^2(G)$ on a Lie group G from a representation-theoretical point of view.

Now, one of the most important and difficult problems in the disintegration theory of group representations is to write down explicit unitary operators that intertwine unitary representations and their decomposition into irreducibles. This is, as a matter of fact, the basic aim of Chap. 3. The answer to this question can help to solve several important problems, for example regarding the study of the solvability of invariant differential operators on homogeneous spaces, or the Frobenius reciprocity problem related to the corresponding multiplicities, the disintegration of generalized invariant vectors, etc. We shall construct intertwining operators of induced representations (sometimes concerning discrete subgroups) and restricted representations of arbitrary nilpotent Lie groups. This will elicit the analysis of the tensor product of two unitary representations. Even in the case of exponential solvable Lie groups these problems are still not fully solved, and we shall only discuss the case of induced representations from normal subgroups.

With the above in mind, we establish the Bonnet-Plancherel formula associated with a monomial representation of a nilpotent Lie group and, more generally, of an exponential solvable Lie group when the inducing subgroup is normal. In Chap. 4, we will also discuss a variant of the Penney-Plancherel formula for induced representations and restrictions.

The orbit method of Chap. 2, which goes back to the early 1970s, asserts that two monomial representations $\pi_i = \mathrm{ind}_{H_i}^G \chi_f (H_i = \exp(\mathfrak{h}_i), 1 \leq i \leq 2)$ of G are irreducible and mutually equivalent. Here $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are two polarizations of $\mathfrak{g}$ at $f \in \mathfrak{g}^*$ that satisfy the Pukanszky condition. We construct an explicit intertwining operator between these representations in the absence of a complete proof of the fact that the product H_1H_2 is a closed subset of G. A third Vergne polarization will come into play and give rise to a unitary intertwining operator. We shall prove a composition theorem involving the Maslov index of the three polarizations. The fundamental results of Auslander-Kostant [3] and Pukanszky [144] about holomorphically induced representations of solvable Lie groups, which hold without any knowledge of real polarizations for a given linear form in the dual Lie algebra, live outside the world of exponential Lie groups. There clearly remains a massive number of open questions to be solved. Chapter 6 will provide the occasion to study certain intertwining operators and real polarizations in the aforementioned rather general setting.

The aim of Chap. 5 is to present two open conjectures concerning certain algebras of invariant differential operators on the space of induced and restricted representations of nilpotent Lie groups. The claim is that whenever these representations have finite multiplicities, the corresponding algebras are isomorphic to the algebra of invariant polynomial functions on some related geometric object (manifold). We will furnish solutions in many special cases and lay out the latest advances in this direction.

Chapter 7 is a sort of generalization of Chaps. 2 and 4 to the setting of exponential groups G. We shall study in every detail the finite and infinite multiplicity of discrete-type monomial representations. Our interest lies in the Penney-Plancherel formula for monomial representations of discrete type, and in a question of Michel Duflo (cf. [54]) regarding the commutativity of the algebra of G-invariant differential operators on line bundles associated with the representation in question. We shall emphasize the result whereby such algebra turns out to be commutative in the case of monomial representations of discrete type and hence gives a counterexample to Duflo's question. The orbit method (cf. [70]) will be a fundamental technique in the study.

The last chapter (Chap. 8) is an application of earlier results of the book. We will characterize bounded, topologically irreducible representations of an exponential solvable Lie group G on Banach spaces by using a triple involving topologically irreducible weighted representations. We shall determine the simple modules of the group algebra $L^1(G)$. These are essentially obtained in the same way as for nilpotent groups, except that one has to generalize the induced representations. This is no longer true for topologically irreducible representations, as shown in [117]. We will show that if the group G is not symmetric, a new type of bounded, irreducible representation of G on Banach spaces crops up. The corresponding representation of $L^1(G)$ is fundamentally different from the induced representation. The construction of such representations is related to the invariant-subspace problem.

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Nomenclature

$\rho(G, H, A, \sigma)$	up-down representation
$\mathscr{E}(G, K)$	the space of continuous functions ξ on G with compact support modulo K
$(\mathscr{H}_\rho^{-\infty})^{K,\lambda}$	space of K-semi-invariant generalized vectors
β_ϕ	Penney distribution
$\exp : \mathfrak{g} \longrightarrow G$	The exponential mapping
λ_G	left regular representation
$\mathbb{C}(\Omega)^K$	field of invariant rational functions
$\mathbb{C}(\Gamma_\tau)^H$	field of invariant rational functions
$\mathbb{C}[\Omega(\pi)]^K$	algebra of invariant polynomial functions
$\mathbb{C}[\Gamma_\tau]^H$	algebra of invariant polynomial functions
$\mathfrak{b}$	real kernel part
$\mathfrak{d}$	real part
$\mathfrak{e}$	complex part
$\mathfrak{g}^*/G$	space of coadjoint orbits
$\mathscr{S}(N)$	the Schwartz-algebra of N
$\mathscr{C}$	composition sequence
$\mathscr{H}(f, \mathfrak{h}, N)$	space of holomorphically induced representation
$\mathscr{H}(f, \eta_f, \mathfrak{h}_i, G)$	space of holomorphically induced representation
$\mathscr{S}$	good sequence of subalgebras
$\mathscr{U}(\mathfrak{g}, \tau)$	infinitesimal invariant operator
ω_U	fundamental weight
$\pi \times \pi'$	outer tensor product of two representations
$\pi_{\mid H}$	a restriction of a representation to a subgroup
$\rho(f, \mathfrak{h}, N)$	holomorphically induced representation
$\rho(f, \eta_f, \mathfrak{h}, G)$	holomorphically induced representation
$\rho(G, H, \pi)$	down-up representation
$\mathrm{ind}_H^G \nu$	induced representation
Δ_G	modular function
Θ	algebra homomorphism
Θ_G	the Kirillov-Bernat mapping

$\widehat{A}^{\mathrm{top}}$	equivalence classes of irreducible Banach space modules of A
$a(\mathfrak{h}_i, \mathfrak{h}_j, \mathfrak{h}_k)$	$= e^{\frac{i\pi}{4}\tau_{ijk}}$
a_ℓ	pointwise Penney distribution
a_π^k	Penney's distribution
$c(i, j, k)$	Maslov index
$D_\pi(G)^K$	algebra of invariant differential operator
$D_\tau(G/H)$	algebra of invariant differential operators
$G \cdot l$	coadjoint orbit
$H \diagdown G \diagup B$	a double coset
$H(x, y)$	Hermitian form
$H_f(x, y)$	Hermitian form
$I_{\mathfrak{h}_2 \mathfrak{h}_1}$	formal intertwining integral
J	complex structure
$K(s, m)$	integral
$P^+ = P^+(f, \mathfrak{n})$	set of positive polarizations
P_X	image function
$R_{a,G/P,\ell}$	retract
$S(e)$	set of jump indices
$S(x, y)$	symmetric form
$Simple(A)$	equivalence classes of simple modules
$T(e)$	set of non-jump indices
$T(f, \eta_f, \mathfrak{h}, G)$	induced representation
$T_{\mathfrak{h}_2 \mathfrak{h}_1}$	intertwining operator
U_e	layer
X^{fin}	finite-rank subspace
$\delta_\ell(s) = \delta_{\ell,\mathscr{S}}(s)$	weight factor
$\mathfrak{a}_\tau$	kernel
$\nu_{\mathfrak{h}}(s)$	action of an automorphism
τ_{ijk}	Maslov index
$\theta_{\mathfrak{h}}(s)$	density
$\Lambda_{\mathfrak{h}}$	intertwining isomorphism

Contents