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Antoine Chambert-Loir

(Mostly) Commutative Algebra

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ISSN 0172-5939 ISSN 2191-6675 (electronic)
Universitext
ISBN 978-3-030-61594-9 ISBN 978-3-030-61595-6 (eBook)
https://doi.org/10.1007/978-3-030-61595-6

Mathematics Subject Classification: 13C, 13B

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Mais je ne m'arrête point à expliquer ceci plus en détail, à cause que je vous ôterais le plaisir de l'apprendre par vous-même, et l'utilité de cultiver votre esprit en vous exerçant...

René Descartes (1596–1659)
La géométrie (1638)

Preface

This book stems out of lectures on commutative algebra given to 4th-year university students at two French universities (Paris and Rennes). At that level, students have already followed a basic course in linear algebra and are essentially fluent with the language of vector spaces over fields. The new topics to be introduced were arithmetic of rings, modules, especially principal ideal rings and the classification of modules over such rings, Galois theory, as well as an introduction to more advanced topics such as homological algebra, tensor products, and algebraic concepts involved in algebraic geometry.

Rewriting the text in view of its publication, I have been led to reorganize or expand many sections, for various reasons which range from giving interesting applications of a given notion, adopting a more natural definition which would be valid in a broader context, to calming the anxiety of the author that "this is not yet enough". For example, the "(Mostly)" of the title refers to the fact that basic notions of rings, ideals, modules are equally important in the context of non-commutative rings — important in the sense that this point of view is definitely fruitful in topics such as representation theory — but was initially motivated by the (conceivably naive) desire of being able to say that a structure of an A-module on an abelian group M is a morphism of rings from A to the ring of endomorphisms of M, the latter ring being non-commutative in general.

This makes the present book quite different from classic textbooks on commutative algebra like Atiyah & Macdonald (1969), Matsumura (1986), Jacobson (1985), or Eisenbud (1995). I am certainly less terse than the first two, and won't go as far as the last three. Doing so, I believe that this book will already be accessible to younger (undergraduate) students, while giving them an idea of advanced topics which are usually not discussed at that level.

This book has also not been written as an absolute treatise (such as the volumes Bourbaki (1989*a*, 2003, 2012) on general algebra and Bourbaki (1989*b*) on commutative algebra). While I have taken the time to explain all basic definitions and elementary examples, its reading probably presupposes some habit of abstract algebra. Sometimes, it even assumes some understanding of parts which are discussed in detail only later, but which certainly were the object of former study by the reader. Consequently, and this is especially true for the first chapters, the reader may be willing to skip some sections for a first reading. The 450 pages of the book don't make it fit for a 15-week course anyway. Another hope is that the reader will enjoy more advanced results of each chapter at the occasion of another visit.

The main protagonists of the book are rings, even *commutative rings*, viewed as generalized numbers in which we have an addition and a multiplication. The first chapter introduces them, as well as the language that is needed to understand their properties. Three ways to forge new rings from old ones are introduced: polynomials, quotients and fractions.

In the second chapter, I study *divisibility* in general rings. As was discovered in the twentieth century, classical properties of integers, such as unique factorization, may not hold anymore and this leads to different attitudes. One can either set up new structures — and this leads to the study of prime or maximal ideals — or restrict oneself to rings that satisfy that familiar properties: then, principal ideal domains, euclidean rings, unique factorization domains come in. First applications to algebraic geometry already appear in this chapter: Hilbert's Nullstellensatz and Bézout's theorem on the number of intersection points of plane curves.

Modules are to rings what vector spaces are to fields. This third chapter thus revisits classic notions of linear algebra in this broader context — operations (intersection, direct sums, products, quotients), linear independence, bases, (multi-)linear forms. . . — but also new notions such as Fitting ideals. Conversely, I have found it interesting to retell the story of vector spaces from this general perspective, establishing the existence of bases and the well-definition of dimension. Nakayama's lemma is also discussed in this chapter, as a tool that sometimes allows one to study modules over general rings by reduction to the case of linear algebra over fields.

The fourth chapter presents *field extensions*. While its culminating point is Galois's theory, we however do not discuss classical "number-theoretical" applications of Galois's theory, such as solvability by radicals or geometric constructions with compass and straight-edge, and for which numerous excellent books already exist. The reader will also observe that the first section of this chapter studies *integral dependence* in the context of rings; this is both motivated by the importance of this notion in later chapter, and by the fact that it emphasizes in a deeper way how linear algebra is used to study algebraic/integral dependence.

The description of modules over fields, that is, of vector spaces, is particularly easy: they are classified by a single invariant, their dimension, which in the finitely generated case, is an integer. The next case where such an explicit description is possible is the object of the fifth chapter, namely the classification of *modules over a principal ideal ring*. This classification can be obtained in various ways and we chose the algorithmic approach allowed for by the Hermite forms of matrices; it also has the interest of furnishing information on the structure of the linear group over euclidean domains. Of course, we give the two classical applications of the general theory, to finitely generated abelian groups (when the principal ideal domain is $\mathbf{Z}$) and to the reduction of endomorphisms (when it is the ring $k[\mathrm{T}]$ of polynomials in one indeterminate over a field k).

Chapter six introduces various tools to study a module over a general ring. Modules of finite *length*, or possessing the *noetherian*, or the *artinian* properties, are analogues of finite-dimensional vector spaces over a field. I then introduce the support of a module, its associated prime ideals, and establish the existence of a primary decomposition. These sections have a more geometric flavour which can be thought of as a first introduction to the algebraic geometry of schemes.

Homological algebra, one of the important concepts invented in the second half of twentieth century, allows one to systematically quantify the defect of injectivity/surjectivity of a linear map, and is now an unavoidable tool in algebra, geometry, topology,. . . The introduction to this topic I provide here goes in various directions. I define projective and injective modules, and use them as a pretext for elaborating on the language of categories, but fall short of defining resolutions and Tor/Ext functors in general.

The next chapter is devoted to the general study of *tensor products*. It is first used to develop the theory of determinants, via the exterior algebra. The general lack of exactness of this operation leads to the definition of a flat module and, from that point, to the study of *faithfully flat descent* and *Galois descent*. This theory, invented by Alexander Grothendieck, formalizes the following question: imagine you need additional parameters to define an object, how can you get rid of them?

Algebras of finite type over a field are the algebraic building blocks of algebraic geometry, and some of the properties studied in this last chapter, Noether's normalization theorem, dimension, codimension, have an obvious geometric content. Algebraically, these rings are also better behaved than general noetherian rings. A final section is devoted to *Dedekind rings*; an algebraic counterparts of curves, they are also very important in number theory, and I conclude the chapter by proving the finiteness of the class groups of number fields.

In the *appendix*, I first summarize general mathematical conventions. The second section proves two important results in *set theory* that are used all over the text: the Cantor–Bernstein theorem and the Zorn theorem. The main part of the book makes use of the language of categories and functors, in a hopefully progressive and pedagogical way, and I have summarized this language in a final section.

Within the text, 11 one-page notes give historical details about the main contributors of the theory exposed in this book, concentrating on their mathematical achievements. By their brevity, they do not replace a serious historical study of the development of the mathematical ideas, and only aim at shining a dim light on these matters. I hope that these short remarks will incite the readers to read books such as Corry (2004), Fine & Rosenberger (1997), Gray (2018), or Bourbaki (1999), or even read the original texts!

Every chapter comes with a large supply of exercises of all kinds (around 300 in total), ranging from a simple application of concepts to additional results that I couldn't include in the text.

Many important concepts and results in commutative algebra were invented or proved in Germany during the first half of the twentieth century, thanks to David Hilbert, Emmy Noether, Wolfgang Krull... For this reason, some theorems still bear a German name, and I have not failed to conform to this tradition regarding Hilbert's *Nullstellensatz* or Krull's *Hauptidealsatz*.

One tradition I tried to step away from, however, is the use of the German alphabet ("fraktur" or "gothic"), for it is sometimes hard to read, and even harder to write by hand. Consequently, prime and maximal ideals, for example, will be denoted by P or M, rather than $\mathfrak{p}$ or $\mathfrak{m}$. The reader will judge for herself or himself whether this shift from typographic tradition was necessary, or even useful. I have however kept German letters for the symmetric group $\mathfrak{S}_n$ and the set $\mathfrak{P}(S)$ of all subsets of a set S.

Statements of the book are numbered as chapter.section.subsection; in the appendix, I have also made use of subsubsections. Enumerations of mathematical statements are of two typographical kinds: when they consist of independent assertions, I have enumerated them by letters (*a*), *b*), *c*),...) while I have enumerated them by roman figures ((i), (ii), (iii),...) when they consisted of related assertions, such as in lists of equivalent properties.

I tried to make the book essentially self-contained, without any need for exterior references. However, I have included in the bibliography some important reference books (textbooks or research books) on the subject, as well as research articles or notes that helped me describe some results, or prepare some exercises.

The final index will hopefully help the reader to track definitions, properties and theorems in the book. I have usually tried to list them by a noun rather by an adjective, so that, for example, "noetherian" is rather to be found at "ring" and "module" (although I have added an entry for this particularly important case).

François Loeser was teaching this course when I was just a beginning assistant professor, and he suggested that I manage the exercises sessions; I thank him for his trust and friendship all over these years. I would also like to thank Sophie Chemla and Laurent Koelblen, who shared their list of exercises with me.

Many thanks are due to Yuri Tschinkel — not only is he a diligent research collaborator, but he is the one who brought the book to a publisher a long time ago.

I also thank Yves Laszlo, Emmanuel Kowalski, Daniel Ferrand and Javier Fresán for their suggestions or encouragements at various times of the elaboration of the book. I also wish to thank Colas Bardavid, Pierre Bernard, Samit Dasgupta, Brandon Gontmacher, Damien Mégy, Ondine Meyer and Andrey Mokhov for pointing out a few unfortunate typos.

I am also grateful to Tom Artin, Alain Combet, Philippe Douroux, Paulo Ribenboim and to the Department of Mathematics at Princeton University for

having graciously allowed me to reproduce some pictures. Although these pictures can be found everywhere on the Internet, I should also thank Elena Griniari, my editor, for having patiently convinced me of the importance of formally requiring these authorizations.

Writing in English is now the daily practice of almost all mathematicians. Nevertheless, this book would have been full of linguistic *bizarreries* without the wonderful work of Barnaby Sheppard, language editor at Springer Nature; I thank him heartily.

I made the final touches to this book during long lockdown months caused by the Covid-19 pandemic, and the following summer. This final work wouldn't have been possible without the availability of an enormous amount of digitized mathematical texts, either behind the closed doors of mathematical publishers (for which my university provides a key), or on public state-funded archives such as the German *Göttinger Digitalisierunszentrum* and the French *Numdam*, as well as on the open websites *Library Genesis* and Alexandra Elbakyan's *Sci-Hub*. Let them all be thanked for their involvement in open science.

Palaiseau, August 2020 *Antoine Chambert-Loir*

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